Raytheon Technologies

The Differential and Riemannian Geometry of Guidance

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Applied Math in Guidance and Control

- **A properly designed guidance algorithm enables vehicles to follow specified paths in space or achieve geometric objectives using feedback control**
- Stability theory, dynamics, and differential geometry are critical to understanding guidance
- Realtime computation is often required, and closed form solutions can be a preferred option for many missions
- Complexity of dynamics will affect guidance performance
- Related references:
  - Greenwood, D.T., Principles of Dynamics, Prentice-Hall, 1987.
  - do Carmo, M.P., Differential Geometry of Curves and Surfaces, Prentice-Hall, 1976.
  - do Carmo, M., Riemannian Geometry, Birkhäuser, 1992.
  - Bullo, F., and A. Lewis, Geometric Control of Mechanical Systems, Springer, New York, 2005.

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Trajectories vs Curves

- In dynamics and control problems, we think of a **state trajectory** as the time evolution of motion resulting from an ordinary differential dynamic equation

$$\text{Dynamic equation: } \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t); \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n,$$

$$\text{Integral flow (solution): } \boldsymbol{x}(t) = \boldsymbol{\Phi}_{\boldsymbol{f}}(\boldsymbol{x}_0, t, t_0)$$

- However, what is equally relevant (and less appreciated) is the role of geometry in the guidance (control) problem and how **differentiable curves** play a role in determining how we create trajectory designs

Various levels of dynamic modeling are used to simulate a missile's motion

Unit Tangent

- Let's consider a time-parametrized curve to see how geometry and physics play a role in guidance
- Given a vehicle with "inertial" position $\boldsymbol{r}(t)$ at time t, the arc length distance formula is given by the integral particle speed $v = \|\dot{\boldsymbol{r}}\| = \left\|\frac{d\boldsymbol{r}}{dt}\right\|$ along the path

$$s_{\boldsymbol{r}}(\boldsymbol{r}(0), \boldsymbol{r}(t)) \equiv \int_0^t \|\dot{\boldsymbol{r}}(\tau)\| \, d\tau = \int_0^t v(\tau) \, d\tau$$

and hence, $v = \dot{s}_{\boldsymbol{r}}(t)$. Assuming the curve $\boldsymbol{r}(t)$ is *regular*, we define the **unit tangent** vector $\mathbf{T}(t)$ at time t as the normalized velocity vector

$$\mathbf{T}(t) = v^{-1} \cdot \dot{\boldsymbol{r}}$$

The unit tangent is the **missile heading**

Total, Tangential, and Normal Acceleration

- The **total acceleration** of the trajectory $\boldsymbol{r}(t)$ is defined as the vector quantity

$$\frac{d\dot{\boldsymbol{r}}}{dt} = \frac{d}{dt}(v \cdot \mathbf{T}) = \dot{v}\,\mathbf{T} + v\,\dot{\mathbf{T}} = \boldsymbol{g} + \boldsymbol{u}$$

Acceleration control

Acceleration due to gravity

where the **tangential acceleration** is given by

$$\boldsymbol{A}_T(t) = \dot{v}\,\mathbf{T}$$

and the **normal (or centripetal) acceleration** is the component orthogonal to $\mathbf{T}$

Id = Identity matrix

$$\boldsymbol{A}_N(t) = v\,\dot{\mathbf{T}} = \left(\mathbf{Id} - \mathbf{T}\,\mathbf{T}^{\mathrm{T}}\right)\frac{d\dot{\boldsymbol{r}}}{dt}$$

- For $\boldsymbol{A}_N(t) \neq 0$, we define the **unit normal** vector as

$$\mathbf{N}(t) = \|\boldsymbol{A}_N(t)\|^{-1}\boldsymbol{A}_N(t)$$

- The **unit binormal vector** of a regular curve with $\boldsymbol{A}_N(t) \neq 0$ is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

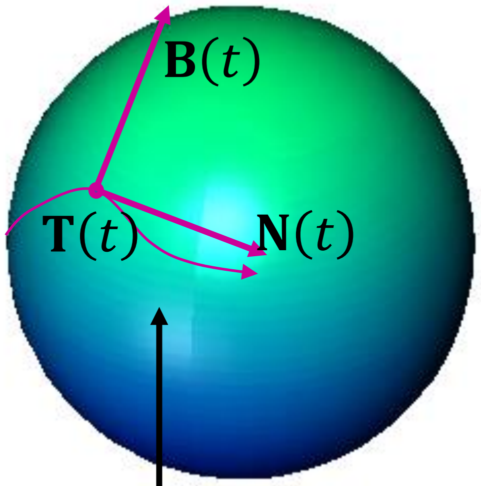
Heading is controlled by calculating desired normal accelerations and converting them to acceleration commands normal to the vehicle body, which are then used by the autopilot

Frenet-Serret Frame and the Sphere Bundle

- Clearly $\mathbf{T}(t)$ belongs to the tangent space at $\boldsymbol{r}(t)$, or $\mathbf{T} \in T_{\boldsymbol{r}}\mathbb{R}^3$. We also have $\mathbf{T}(t) \in S_{\boldsymbol{r}(t)}\mathbb{R}^3$, where

$$S_{\boldsymbol{r}}\mathbb{R}^3 = \left\{\mathbf{Y} \in T_{\boldsymbol{r}}\mathbb{R}^3 | \langle \mathbf{Y}, \mathbf{Y} \rangle = 1\right\} \subset T_{\boldsymbol{r}}\mathbb{R}^3$$

  and $S\mathbb{R}^3 \equiv \bigcup_{\boldsymbol{r} \in \mathbb{R}^3} S_{\boldsymbol{r}}\mathbb{R}^3$ is the sphere bundle for $\mathbb{R}^3$
- The Frenet-Serret frame $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ [Greenwood] can be mapped into the sphere bundle and its tangent space $TS\mathbb{R}^3$
  - The unit vector $\mathbf{T}(t)$ defines the tangent direction at time t (the point on the sphere)
  - The unit vector $\mathbf{N}(t)$ defines the direction of acceleration of $\mathbf{T}(t)$
  - The unit vector $\mathbf{B}(t)$ creates the right-handed system $\left(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\right)$
  - Explicit computation of the tangent space basis $(\mathbf{N}(t), \mathbf{B}(t))$ is not necessary in the guidance design process



The sphere bundle is the natural space for missile guidance design

Linear Feedback/Feedforward Control System Example

- Consider the simple dynamic system with fully-controllable acceleration

$$\ddot{\boldsymbol{r}} = \boldsymbol{g} + \boldsymbol{u}$$

- Assuming $\dot{\boldsymbol{r}}_\delta$ is a desired velocity with $\dot{\boldsymbol{r}}, \dot{\boldsymbol{r}}_\delta, \ddot{\boldsymbol{r}}_\delta$ available for control, the Euclidean feedback/feedforward control

$$\boldsymbol{u} = -\boldsymbol{g} - c(\dot{\boldsymbol{r}} - \dot{\boldsymbol{r}}_\delta) + \ddot{\boldsymbol{r}}_\delta$$

  produces the first-order error dynamics for $\boldsymbol{e} = \dot{\boldsymbol{r}} - \dot{\boldsymbol{r}}_\delta$:

$$\dot{\boldsymbol{e}} = -c\boldsymbol{e}$$

- **Main Issue and Resolution:** Cannot usually control missile thrust, but we can control acceleration normal to the unit heading $\mathbf{T} \in S^2$ indirectly through control of acceleration normal to the vehicle body

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Simple Guidance on the Sphere

- For simplicity, we equate $S_{\boldsymbol{r}}\mathbb{R}^3$ with S^2. Assume the normal acceleration is determined by the projection of acceleration due to gravity $\boldsymbol{g}$ and controlled by a vector input $\boldsymbol{u} \in T_{\mathbf{T}}S^2$ in the tangent space

$$\boldsymbol{A}_N(t) = v\,\dot{\mathbf{T}} = \left(\mathbf{Id} - \mathbf{T}\,\mathbf{T}^{\mathrm{T}}\right)\boldsymbol{g} + \boldsymbol{u}$$

The tangent space $T_{\mathbf{T}}S^2$ is the natural space for simple guidance design

and assume the vehicle speed never approaches zero, or $v \geq \varepsilon > 0$.

- We want to develop a guidance design that ensures the vehicle unit heading $\mathbf{T}$ converges to a desired unit heading $\mathbf{T}_\delta$
- Several possible objectives for guidance
  - Solve a possibly time-varying boundary value problem (intercept)
  - Converge to a desired path (midcourse trajectory tracking)
  - Remain in an invariant set (safety, stability)

Nature and the Mathematics of Guidance Design

- Geometric principles can be used to derive simple control commands called **steering laws**
- Predators evolved to take advantage of these principles

Tiger Beetle

Peregrine Falcon

- A predator needs a **regulator**, or control system, that takes in measurements to produce normal acceleration and stabilize the trajectory relative to a steering law

Pursuit Steering Law (Tiger Beetle)

- Let $\boldsymbol{r}$ be the predator position and $\boldsymbol{r}_T$ be the prey position, the line of sight (LOS) unit vector from the predator to the prey is

$$\mathbf{T}_{\text{LOS}} = \text{vers}(\boldsymbol{r}_T - \boldsymbol{r}) = \frac{\boldsymbol{r}_T - \boldsymbol{r}}{\|\boldsymbol{r}_T - \boldsymbol{r}\|}$$

- Ideal (Virtual) Interceptor Velocity for Pursuit Steering $(v = \|\dot{\boldsymbol{r}}\|)$:

$$\dot{\boldsymbol{r}}_\delta = v\,\mathbf{T}_\delta = v\,\mathbf{T}_{\text{LOS}} = v\,\text{vers}(\boldsymbol{r}_T - \boldsymbol{r})$$

- Ideal Interceptor Acceleration for Pursuit Steering:

$$\begin{aligned}\ddot{\boldsymbol{r}}_\delta = \dot{v}\,\mathbf{T}_\delta + v\,\dot{\mathbf{T}}_\delta &= \dot{v}\,\mathbf{T}_{\text{LOS}} + v\,\dot{\mathbf{T}}_{\text{LOS}} \\ &= \dot{v}\,\mathbf{T}_{\text{LOS}} + \boldsymbol{\omega}_{\text{LOS}/I} \times (v\,\mathbf{T}_{\text{LOS}}) \\ &= \dot{v}\,v^{-1}\dot{\boldsymbol{r}}_\delta + \boldsymbol{\omega}_{\text{LOS}/I} \times \dot{\boldsymbol{r}}_\delta\end{aligned}$$

where $\boldsymbol{\omega}_{\text{LOS}/I}$ is the angular velocity of the LOS relative to inertial (we assume that $\boldsymbol{\omega}_{\text{LOS}/I}^{\text{T}} \cdot \mathbf{T}_{\text{LOS}} = 0$)

$\mathbf{T}_\delta$ and $\dot{\mathbf{T}}_\delta$ are steering commands in a pursuit guidance design

Feedback/Feedforward Regulation

- Given an ideal steering law, how does one develop a control (regulator) to allow for convergence to the ideal?
  - One can use Lyapunov-based stability methods to derive a stabilizing control law relative to a desired trajectory
  - Geometric methods can be used to derive coordinate-free control
- Feedback control – a control function of the dynamic state and desired command
- Feedforward control – a function of the highest derivative of the command that produces the error system of differential equations

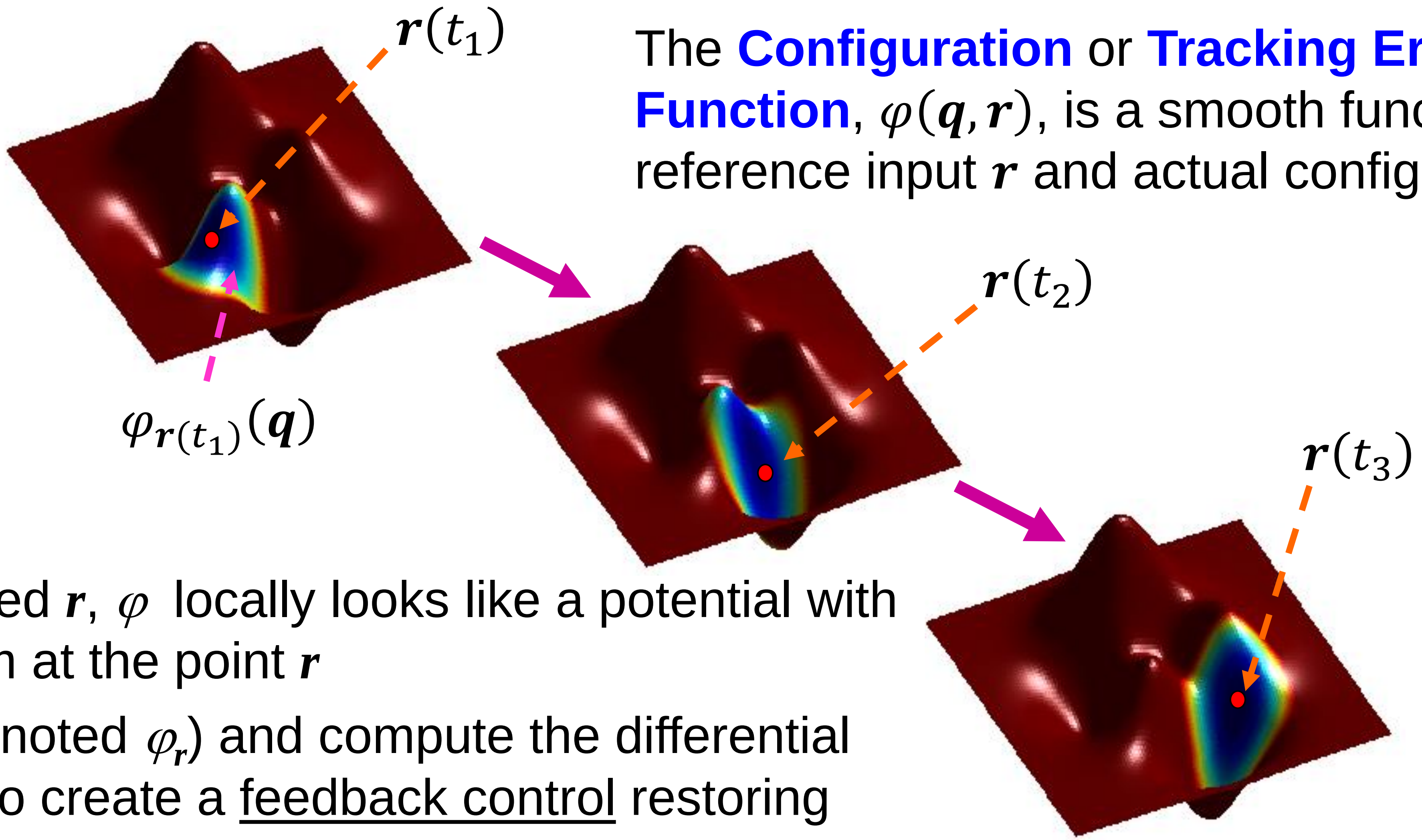
Ensuring stability is an important part of any control design process

Aleksandr Mikhailovich Lyapunov

Born	June 6, 1857 Yaroslavl, Russian Empire
Died	November 3, 1918 (aged 61) Odessa, Ukrainian People's Republic
Nationality	Russian

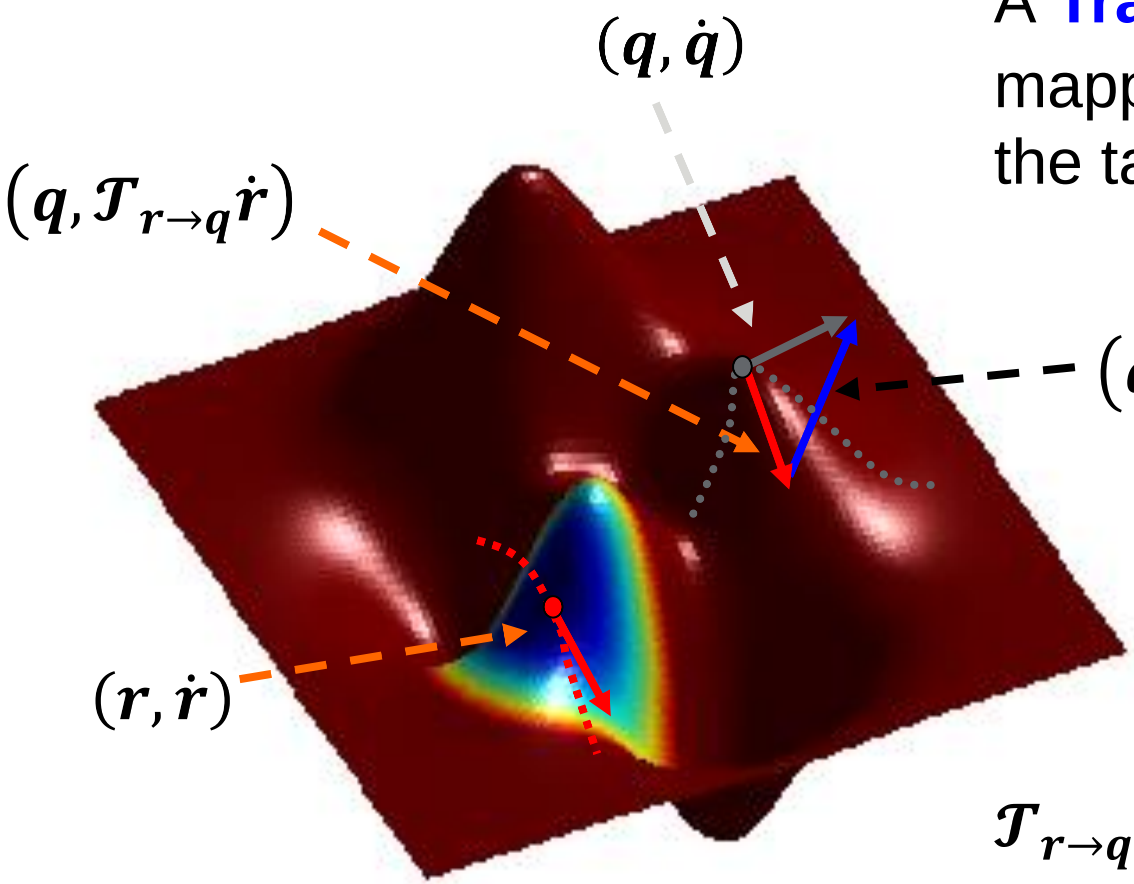
Configuration Error Function Concept [Bullo]

The **Configuration** or **Tracking Error Function**, $\varphi(\boldsymbol{q},\boldsymbol{r})$, is a smooth function of reference input $\boldsymbol{r}$ and actual configuration $\boldsymbol{q}$



- For a fixed $\boldsymbol{r}$, φ locally looks like a potential with minimum at the point $\boldsymbol{r}$
- Fix $\boldsymbol{r}$ (denoted φ_r) and compute the differential $d\varphi_{\boldsymbol{r}}(\boldsymbol{q})$ to create a <u>feedback control</u> restoring force

Vector Error & Transport Concept [Bullo]



A **Transport Mapping** $\mathcal{T}_{r \to q}$ is a smooth mapping that carries a tangent vector from the tangent space of r to that of q

$$(q, \underbrace{\dot{q} - \mathcal{T}_{r \to q} \dot{r}}_{\dot{e} \,=\, \text{Vector Error}})$$

$\mathcal{T}_{r \to q}$ will be used to transport the desired acceleration command for feedforward control

Geometric Midcourse Design

- **Steering Law**
  - Specifies the ideal direction to move the interceptor given a steering condition
  - Relative 3-D information
  - Can be a function of the target, the missile speed, predictions, etc.
  - Must be differentiable with bounded derivative as convergence is proven in infinite time (no discontinuities)

$\mathbf{T}_\delta$	Ideal, or desired, heading
$\dot{\mathbf{T}}_\delta$	Time-derivative of the Ideal heading

- **Regulation**
  - Specifies control design to regulate to the ideal direction and orthogonal to the heading vector
  - Designed on the sphere to obtain near global stability

Gravity compensation

$$\boldsymbol{u} = -\left(\mathbf{Id} - \mathbf{T}\,\mathbf{T}^{\mathrm{T}}\right)\boldsymbol{g} + v\left(\mathcal{T}_{\mathbf{T}_\delta \to \mathbf{T}}\dot{\mathbf{T}}_\delta - \partial\varphi_{\mathbf{T}_\delta}(\mathbf{T})\right)$$

Transported ideal curvature

Gradient of config error function between desired and actual heading

The Spherical Geometry of the Guidance Problem

Can choose a potential function between desired and missile tangent unit directions

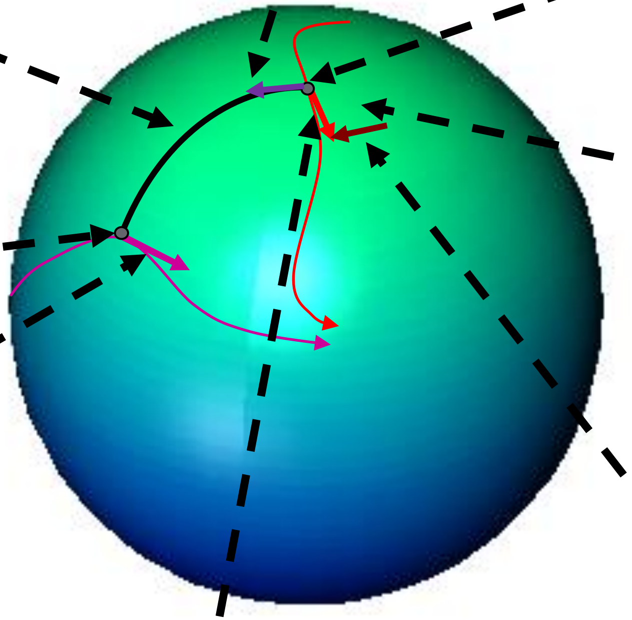
$\varphi(\mathbf{T}, \mathbf{T}_\delta)$

Negative gradient of the distance function

$-\partial\varphi_{\mathbf{T}_\delta}(\mathbf{T})$

Unit mag curve represents interceptor velocity direction as a function of time

$\mathbf{T}(t)$



Desired intercept velocity dir path defined by steering law

$\mathbf{T}_\delta(t)$

Transport of curvature

$\mathcal{T}_{\mathbf{T}_\delta \to \mathbf{T}} \dot{\mathbf{T}}_\delta$

Steering law curvature

$\dot{\mathbf{T}}_\delta$

Curvature error is a stable first order system

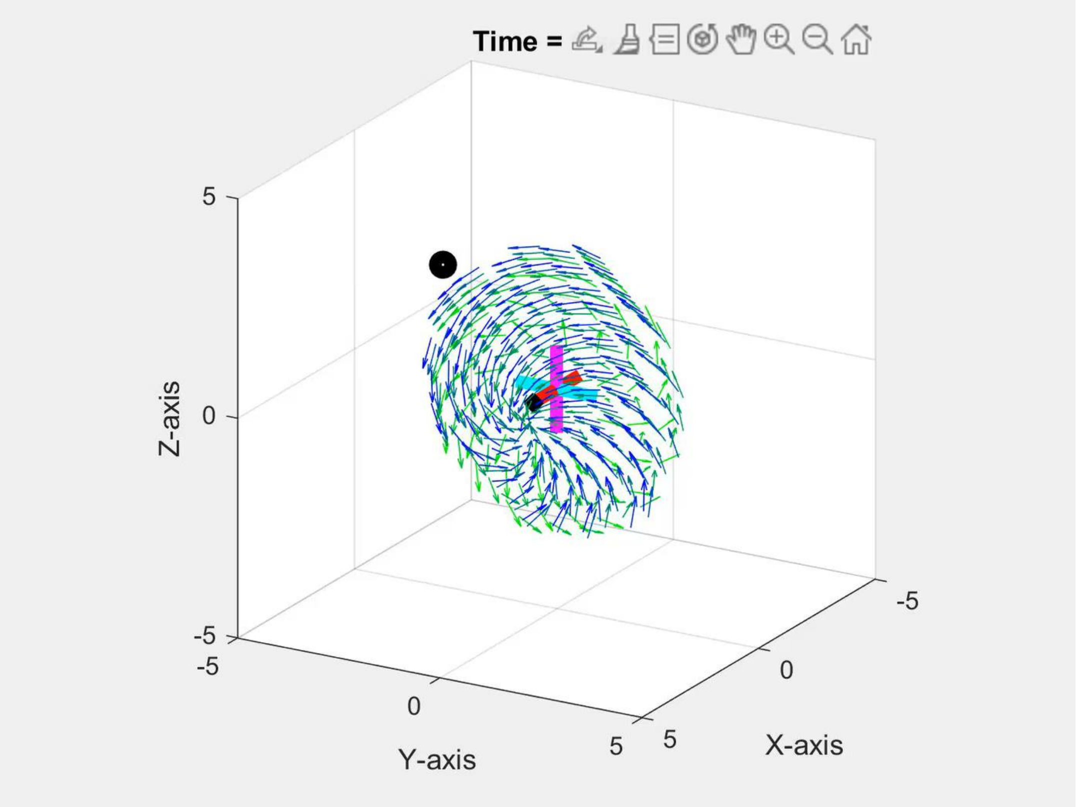
$\dot{\boldsymbol{e}} = \mathbf{T} - \gamma_{\mathbf{T}_\delta \to \mathbf{T}_\beta} \dot{\mathbf{T}}_\delta = -\partial\varphi_{\mathbf{T}_\delta}(\mathbf{T})$

Interceptor curvature

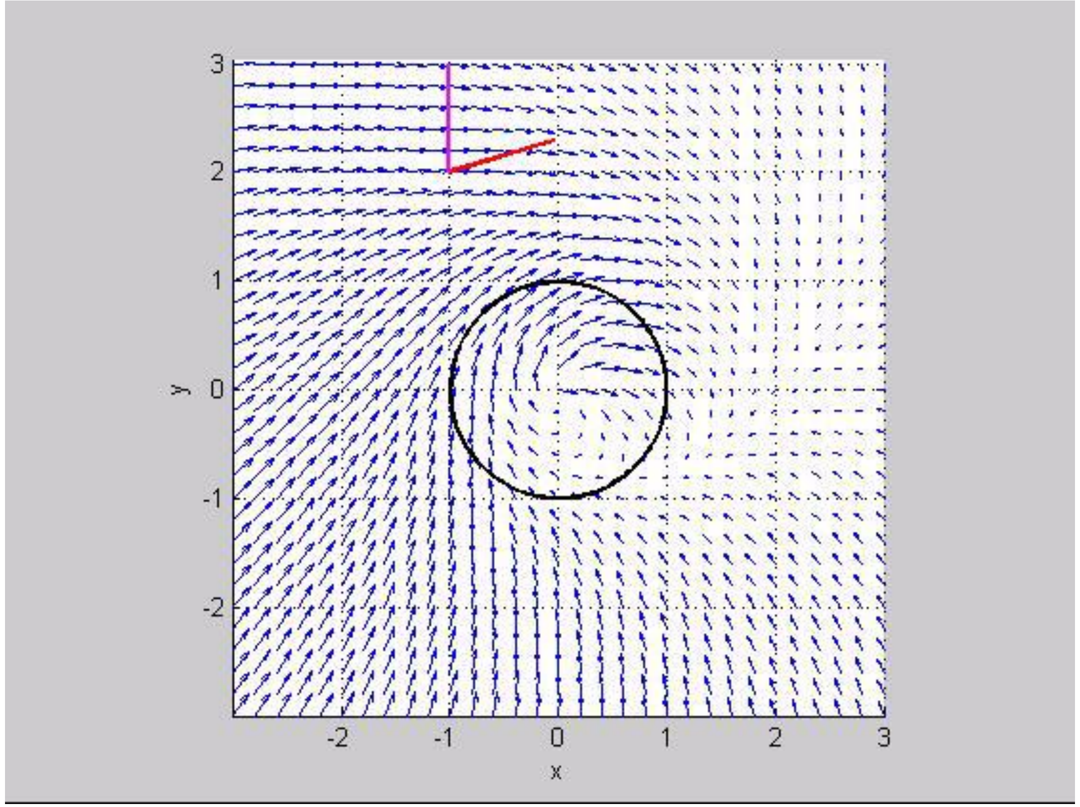
$\dot{\mathbf{T}}$

Practical Geometric Guidance Solutions

Time-varying Docking Problem



Moving Loiter



BACKUP

Frenet-Serret Frame and Radius of Curvature

- The pair $(\mathbf{T}, \mathbf{N})$, when defined, is a basis for the **osculating plane** and $(\mathbf{T}, \mathbf{N}, \mathbf{B})$, when defined, forms a right-handed orthonormal basis called the **Frenet-Serret Frame**
- Let $\omega_{\mathbf{B}} = \langle \dot{\mathbf{T}}, \mathbf{N} \rangle$. We have

$$\frac{d\mathbf{T}}{dt} = \omega_{\mathbf{B}}\mathbf{N} = (\omega_{\mathbf{B}}\mathbf{B}) \times \mathbf{T}\,.$$

- When defined, the **radius of curvature** $\rho(t)$ at time t is

$$\omega_{\mathbf{B}}(t) = \big(\rho(t)\big)^{-1} v(t).$$

- The **center of curvature** vector at time t is given by

$$\boldsymbol{C}(t) = \rho(t)\,\mathbf{N}(t).$$

Minimum radius of curvature is inversely related to the G-limit of missile

Dynamics of the Osculating Plane

- Differentiating the unit normal vector, we have

$$\frac{d\mathbf{N}}{dt} = \frac{d}{dt}(\mathbf{B} \times \mathbf{T}) = \frac{d\mathbf{B}}{dt} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{dt}$$
$$= \frac{d\mathbf{B}}{dt} \times \mathbf{T} + (\omega_{\mathbf{B}}\mathbf{B}) \times \mathbf{N} = \frac{d\mathbf{B}}{dt} \times \mathbf{T} - \omega_{\mathbf{B}}\mathbf{T}$$

- Since we have

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{T}}{dt} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{dt} = \mathbf{T} \times \frac{d\mathbf{N}}{dt}$$

$\frac{d\mathbf{B}}{dt}$ must be parallel to $\mathbf{N}(t)$, so we can write

$$\frac{d\mathbf{B}}{dt} = -\omega_{\mathbf{T}}\mathbf{N} = (\omega_{\mathbf{T}}\mathbf{T}) \times \mathbf{B}$$

$\omega_{\mathbf{T}}$ is related to the missile **torsion**. If $\omega_{\mathbf{T}} = 0$, the motion of the missile lies in a plane

where $\omega_{\mathbf{T}} = -\left\langle \frac{d\mathbf{B}}{dt}, \mathbf{N}(t) \right\rangle$ is the **rotation rate of the osculating plane**.

Dynamics of the Frenet-Serret Frame

- Thus,

$$\frac{d\mathbf{N}}{dt} = -\omega_{\mathbf{T}}\mathbf{N} \times \mathbf{T} - \omega_{\mathbf{B}}\mathbf{T}$$
$$= \omega_{\mathbf{T}}\mathbf{B} - \omega_{\mathbf{B}}\mathbf{T}$$

- The dynamics of the Frenet-Serret frame are given by

$$\frac{d\mathbf{T}}{dt} = (\omega_{\mathbf{B}}\mathbf{B}) \times \mathbf{T}$$
$$\frac{d\mathbf{N}}{dt} = (\omega_{\mathbf{T}}\mathbf{T}) \times \mathbf{N} + (\omega_{\mathbf{B}}\mathbf{B}) \times \mathbf{N}$$
$$\frac{d\mathbf{B}}{dt} = (\omega_{\mathbf{T}}\mathbf{T}) \times \mathbf{B}$$

In missile guidance design, the speed and time history of the trajectory are of primary importance (no unit-speed normalization required)

- Where the **Darboux Angular Velocity** is defined by $\boldsymbol{\omega}(t) = \omega_{\mathbf{T}} \cdot \mathbf{T}(t) + \omega_{\mathbf{B}} \cdot \mathbf{B}(t)$

Configuration Error Definition [Bullo]

- **Definition:** Let $\boldsymbol{r} \in M$ and $\boldsymbol{q} \in M$ denote respectively the reference and controlled configurations. A smooth, symmetric (wrt interchanging of input variables) function $\varphi: M \times M \to \mathbb{R}$ is a **configuration error function** if for each $\boldsymbol{r} \in M$, $\varphi_{\boldsymbol{r}}(\boldsymbol{q})$ is proper, bounded from below, and φ satisfies

1. $\varphi(\boldsymbol{r}, \boldsymbol{r}) = 0,$
2. $d\varphi_{\boldsymbol{r}}(\boldsymbol{q})|_{\boldsymbol{q}=\boldsymbol{r}} = 0,$
3. $\mathrm{Hess}\ \varphi_{\boldsymbol{r}}(\boldsymbol{q})|_{\boldsymbol{q}=\boldsymbol{r}}$ is positive definite.

This configuration error function will serve as a Lyapunov function to demonstrate stability

Transport Mapping Definition [Bullo]

- A **transport mapping** is a smooth bitensor field $\boldsymbol{\mathcal{T}}$ on $M \times M$ satisfying

1. $\boldsymbol{\mathcal{T}}_{\boldsymbol{r \to q}} \in GL(T_{\boldsymbol{r}}M, T_{\boldsymbol{q}}M)$, that is, $\boldsymbol{\mathcal{T}}_{\boldsymbol{r \to q}}$ works like a smooth matrix that maps vectors from the tangent space of $\boldsymbol{r}$ to the tangent space of $\boldsymbol{q}$, and
2. $\boldsymbol{\mathcal{T}}_{\boldsymbol{q \to q}} = \mathbf{Id}$.

where $\boldsymbol{\mathcal{T}}_{\boldsymbol{r \to q}}$ denotes the evaluation of $\boldsymbol{\mathcal{T}}$ at $\boldsymbol{p} = (\boldsymbol{r}, \boldsymbol{q})$.

- Parallel Transport is an example of a transport mapping, but transport is a more general concept

The transport mapping will be used to create a vector error between tangent vectors at different points on the sphere

Compatibility Condition

- Configuration Error and Transport Mapping should be consistent with each other to ensure stability of a closed loop control
- Cannot expect stability if one chooses two inconsistent methods of configuration and vector error
- The **compatibility condition** is equivalent to the condition that

$$\dot{\varphi}(\boldsymbol{q},\boldsymbol{r}) = d\varphi_{\boldsymbol{r}}(\boldsymbol{q})\big(\dot{\boldsymbol{q}} - \mathcal{T}_{\boldsymbol{r}\to\boldsymbol{q}}\dot{\boldsymbol{r}}\big)$$ (Lemma 11.16 [Bullo])

Differential holding $\boldsymbol{r}$ fixed

- Euclidean Example: Using $\varphi(\boldsymbol{q},\boldsymbol{r}) = \frac{1}{2}k(\boldsymbol{q}-\boldsymbol{r})^{\top}(\boldsymbol{q}-\boldsymbol{r})$, we have

$$\begin{aligned}\dot{\varphi}(\boldsymbol{q},\boldsymbol{r}) &= k(\boldsymbol{q}-\boldsymbol{r})^{\top}(\dot{\boldsymbol{q}}-\dot{\boldsymbol{r}}) \\ &= \langle k(\boldsymbol{q}-\boldsymbol{r}), \dot{\boldsymbol{q}} - \mathbf{Id}\,(\dot{\boldsymbol{r}})\rangle \\ &= \big\langle \partial\varphi_{\boldsymbol{r}}(\boldsymbol{q}), \dot{\boldsymbol{q}} - \mathcal{T}_{\boldsymbol{r}\to\boldsymbol{q}}\dot{\boldsymbol{r}}\big\rangle = d\varphi_{\boldsymbol{r}}(\boldsymbol{q})\big(\dot{\boldsymbol{q}} - \mathcal{T}_{\boldsymbol{r}\to\boldsymbol{q}}\dot{\boldsymbol{r}}\big)\end{aligned}$$

S^2 Configuration Error/Transport Mapping for Guidance

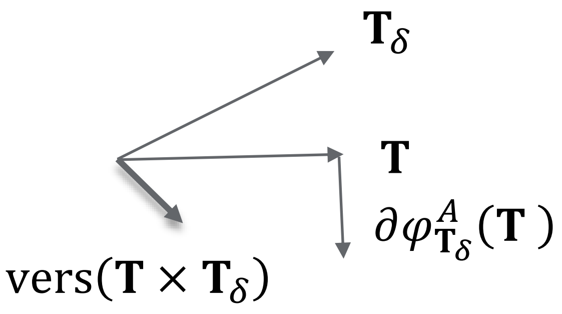
Physical Pendulum Prototype

Direction of max increase is pointing away from the vector $\mathbf{T}_\delta$ and in the tangent plane of $\mathbf{T}$

- Configuration Error Function:

$$\varphi^A(\mathbf{T}, \mathbf{T}_\delta) = k_p(1 - \langle \mathbf{T}, \mathbf{T}_\delta \rangle)$$

- Gradient wrt $\mathbf{T}$ holding $\mathbf{T}_\delta$ fixed:



$$\partial\varphi^A_{\mathbf{T}_\delta}(\mathbf{T}) = -k_p\left(\mathbf{Id} - \mathbf{T}\mathbf{T}^\top\right)\mathbf{T}_\delta = k_p([\mathbf{T}]^\times)^2\mathbf{T}_\delta$$
$$= -k_p \sin(\arccos\langle \mathbf{T}, \mathbf{T}_\delta \rangle)\,(\mathrm{vers}(\mathbf{T} \times \mathbf{T}_\delta) \times \mathbf{T})$$

- Transport Mapping:

$$\mathcal{T}^A_{\mathbf{T}_\delta \to \mathbf{T}} = \langle \mathbf{T}, \mathbf{T}_\delta \rangle\, \mathbf{Id} + [\mathbf{T}_\delta \times \mathbf{T}]^\times$$

Skew-symmetry operator

Thank you.